

General Deformations of Point Configurations Viewed By a Pinhole Model Camera

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Abstract

This paper is a theoretical study of the following Non-Rigid Structure from Motion problem. What can be computed from a monocular view of a parametrically deforming set of points? We treat various variations of this problem for affine and polynomial deformations with calibrated and uncalibrated cameras. We show that in general at least three images with quasi-identical two deformations are needed in order to have a finite set of solutions of the points' structure and calculate some simple examples.

1 Introduction

Non-Rigid Structure from Motion (NRSfM) from a monocular camera has been addressed in a number of papers. The setup is a single camera (monocular) tracking a deforming object. Of course if there are no constraints every point in every image has an unbounded depth ambiguity. In the literature various constraints have been added, locally or globally, to facilitate the computation of numerical solutions of the 3D shape of a deforming object.

As the general problem is very unconstrained there have been many papers addressing certain specifications of the general case, for example instead of a perspective camera a weak perspective or affine camera is assumed, [1, 6, 15, 3, 2] or instead of a general deformation of the object the assumption is that the nonrigid object deforms as a linear combination of K rigid shapes [10]. There are also a number of papers that constrain the deformation of the object to a physical model or a parameterized family of deformations

which they then attempt to solve for in an optimization framework [14]. A review of much of the relevant literature can be found in [21].

The general case of deforming configurations of points has also received attention, but with some restrictions. Some authors consider configurations of points moving with constrained motions [26, 18]. Other papers treat general motion but restrict their analysis to the case of a single point [22, 16, 17]. There has not been much published on the theoretical underpinnings of the recovery of structure of deforming configurations of points.

In this paper we analyze, for the first time, the complexity and ambiguities of a fixed perspective camera tracking a parametrically set of deforming body of points in 3D. When the points move rigidly this is the classic Structure from Motion problem [11].

The specific deformations we analyze are affine and more generally polynomial. We show that when the camera is calibrated and the body undergoes an affine deformation, a matching constraint similar to the classical epipolar geometry can be formulated. We show that from two images only, one cannot recover neither the deformation nor the original points. When three images (i.e. two deformations) are available, we show that in a generic situation, the remaining ambiguity is still three-dimensional. However, when the two deformations are quasi-identical (see below for a complete definition), we show that it is exactly one answer.

Also we show that an invariant shape description can be recovered from 3 images, i.e. from a first view and two other views after deformations. The recovery of this invariant does not require calibration.

Then we turn our attention to the case of complete reconstruction (deformation and structure) for general polynomial deformations. We show that it can be calculated, from a calibrated camera and 3 images, i.e. from the first view and a two other images coming from the same deformation repeated twice. We also show that the number of points needed to be tracked depends on the number of parameters of the deformation.

The case of a full 3D projective transformations is not doable as the deformation can always be explained by an affine transformation;

$$[I; 0] \begin{pmatrix} A_{3 \times 4} \\ a \ b \ c \ d \end{pmatrix} = [I; 0] \begin{pmatrix} A \\ 0 \ 0 \ 0 \ 1 \end{pmatrix}$$

We are mainly interested in the theoretical possibilities both as to number of corresponding points and number of images needed. We present algebraic solutions and do not treat the numerics and instabilities of these problems.

2 Affine Deformations

We start with the study of point correspondences under affine deformations. This has both a practical and a theoretical impact. On the practical side, we shall see that one can write the matching constraint in a similar manner to the classical epipolar geometry. Thus, finding correspondences between two images of an affinely deforming configuration of points can essentially be done with the same machinery as that for the case of classical images of

rigidly moving bodies. Of course, we consider only invertible affine deformations. This choice is motivated to avoid degenerated situations, where two distinct points collapse to a single point after the deformation.

On the theoretical front, we shall use this result extensively in the sequel.

2.1 Essential Matrix

Let us consider a set of deforming points being imaged with a calibrated camera, which can be assumed to be $[I; 0]$. The projection of a point P in homogeneous coordinates in the first image is in homogeneous coordinates $[I; 0]P$, while the projection in the second image is in homogeneous coordinates $[I; 0] \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} P$. Eliminating P from these two sets of equations leads to a bilinear constraint over the corresponding image points q, q' , that can be written through a so-called *essential matrix* E : $q'^t E q = 0$.

Lemma 1. *The essential matrix of this pair of images is: $E \equiv [t]_{\times} A$, where $[t]_{\times}$ is the matrix of the cross product with t in the standard basis of \mathbb{R}^3 .*

Proof. Let us consider a point P in \mathbb{P}^3 not at infinity and projected to q in the first image. Then $q \equiv [I; 0]P$. Thus $P = [\lambda q, 1]^t$, for some $\lambda \in \mathbb{R}$. It is projected in to the second image as $q' \equiv [A; t]P$. Thus $q' \equiv (\lambda Aq + t)$. Then $[t]_{\times} q' \equiv [t]_{\times} Aq$, where $[t]_{\times}$ is the matrix of $a \mapsto t \wedge a$, in the standard basis of \mathbb{R}^3 , with \wedge being the cross product. Since $[t]_{\times}$ is a skew-symmetric matrix, this yields $q'^t [t]_{\times} Aq = 0$. Thus $E \equiv [t]_{\times} A$. \square

We denote by \equiv equality modulo multiplication by a non-zero scalar.

If there are matching pairs of points between the two images: $(q_i, q'_i)_{i=1, \dots, n}$, the following equations hold:

$$q_i'^t E q_i = 0 \quad (1)$$

for each i and rank E is 2 as rank $[t]_{\times}$ is 2. Indeed the deformation is assumed to be invertible, i.e. $\det(A) \neq 0$. E can thus be computed from 7 pairs of corresponding points in general position or linearly from at least 8 pairs of points.

2.2 Deformation Recovery

Once the essential matrix $E \equiv [t]_{\times} A$ is computed, deformation can be recovered up to a 4-parameter ambiguity. To show this, we recall a lemma from [11] (page 255): If a rank 2 matrix F can be decomposed in two different ways as $F = [t]_{\times} A = [\tilde{t}]_{\times} \tilde{A}$ then there exists a constant $\lambda \neq 0$ and $v \in \mathbb{R}^3$, such that: $\tilde{t} = \lambda t$ and $\lambda \tilde{A} = A + tv^t$.

Can more than two images help? Let us consider the situation where the deformation between the first and the second image is (A, a) and between the second and the third (B, b) . We have now three distinct essential matrices: $E_{12} \equiv [a]_{\times} A$, $E_{23} \equiv [b]_{\times} B$ and $E_{13} \equiv [Ba + b]_{\times} BA$.

From E_{12} , we can compute a_0 and A_0 such that $\exists \alpha \neq 0, a = \alpha a_0$ and $\exists v_1, A = \frac{1}{\alpha}(A_0 + a_0 v_1^t)$. From E_{23} , we can compute b_0 and B_0 such that $\exists \beta \neq 0, b = \beta b_0$ and

$\exists v_2, B = \frac{1}{\beta}(B_0 + b_0 v_2^t)$. From the third essential matrix E_{13} we can compute c_0 and C_0 such that:

$$\exists \gamma \neq 0, Ba + b = \gamma c_0 \quad (2)$$

and

$$\exists v_3, BA = \frac{1}{\gamma}(C_0 + c_0 v_3^t) \quad (3)$$

From equations (2) and (3), we get the following system:

$$\begin{cases} \alpha(B_0 + b_0 v_2^t)a_0 + \beta^2 b_0 - \beta \gamma c_0 & = 0 \\ \gamma(B_0 + b_0 v_2^t)(A_0 + a_0 v_1^t) - \alpha \beta (C_0 + c_0 v_3^t) & = 0 \end{cases} \quad (4)$$

Furthermore one has to enforce the that none of α, β nor γ vanishes. Formally, this is equivalent to computing in the localization of the polynomial ring with respect to these variables [7]. Concretely, one must introduce new variables: x, y, z and the equations: $\alpha x - 1 = \beta y - 1 = \gamma z - 1 = 0$.

Eventually the number of unknowns is $N = 9 + 3 + 3 = 15$ and we have exactly 15 equations. They define an algebraic sub-variety X of \mathbb{A}^{15} (see [13, 8]), where we denote by \mathbb{A}^k the k -dimensional affine space. Now the task is to compute the dimension of X . Our data is expressed as real numbers. However nothing prevents us from viewing the variety X as defined over the field of complex numbers. This does not alter its algebraic dimension (see for instance exercise 3.20 in [12]). Moreover \mathbb{C} is an algebraically closed field. This allows one to benefit from algebraic geometry theorems.

Indeed our concern now is to determine the dimension of X . If it is zero, this means that X is a finite set. In this case, one would be interested in its degree in order to estimate the number of solutions. However if the dimension is positive, there are infinitely many solutions. A word of caution is necessary here. While the algebraic dimension is the same over the reals or the complex numbers, the topological dimension can vary. When the dimension is zero, or in other words when the variety is a finite set, this property remains unchanged over reals or the complex numbers. When the dimension is strictly positive, it might be that the set of real solutions is finite or even empty. In our setting, we will see that when the algebraic dimension is strictly positive, the set of real solutions is also infinite.

We prove that indeed X has strictly positive dimension.

Theorem 1. *For two unrelated deformations, the dimension of X is always greater or equal to 3. Moreover for generic deformations, it is precisely 3.*

Proof. Let $\alpha_0, \beta_0, \gamma_0, x_0, y_0, z_0, v_{10}, v_{20}, v_{30}$ be a point on X . Let $a = \alpha_0 a_0, A = 1/\alpha_0(A_0 + a_0 v_{10}^t), b = \beta_0 a_0, B = 1/\beta_0(B_0 + b_0 v_{20}^t)$. Then we know that $Ba + b = \gamma_0 c_0$ and $BA = 1/\gamma_0(C_0 + c_0 v_{30}^t)$. Consider the variety Y defined by the following system:

$$\begin{cases} \alpha'(B + b v_2^t)a + \beta'^2 b - \beta' \gamma'(Ba + b) & = 0 \\ \gamma'(B + b v_2^t)(A + a v_1^t) - \alpha' \beta'(BA + (Ba + b)v_3^t) & = 0 \\ \alpha' x' - 1 & = 0 \\ \beta' y' - 1 & = 0 \\ \gamma' z' - 1 & = 0 \end{cases}$$

The varieties X and Y are easily seen to be isomorphic by the linear mapping:

$$\begin{aligned} (\alpha, \beta, \gamma, x, y, z, v_1^t, v_2^t, v_3^t) &\mapsto \\ (\alpha', \beta', \gamma', x', y', z', v_1^{tt}, v_2^{tt}, v_3^{tt}) &= \\ (\alpha/\alpha_0, \beta/\beta_0, \gamma/\gamma_0, \alpha_0 x, \beta_0 y, \gamma_0 z, 1/\alpha_0^2(v_1^t - v_{10}^t), 1/\beta_0^2(v_2^t - v_{20}^t), 1/\gamma_0^2(v_3^t - v_{30}^t)) \end{aligned}$$

Therefore $\dim(X) = \dim(Y)$.

The variety Y depends on the deformations (A, a) and (B, b) . One can define a family of varieties, whose fiber at (A, a, B, b) is precisely the variety Y . To proceed, we shall define $D \cong \mathbb{A}^{12}$ to be the space of affine deformations. Then consider the variety $Z \subset \mathbb{A}^{15} \times (D \times D)$ defined by the above system, where A, a, B, b are also considered as variables. Then the projection $\pi_2 : Z \rightarrow D \times D$ defines a family of varieties whose fibers are precisely the different varieties Y .

The variety Z is embedded into $\mathbb{A}^{15} \times D \times D = \mathbb{A}^{39}$ and is defined by 15 equations. Therefore we have: $\dim(Z) \geq 39 - 15 = 24$.

The projection π_2 is surjective as easily seen from the system. For each pair of deformations (A, a) and (B, b) , the point defined by $\alpha = \beta = \gamma = x = y = z = 1$ and $v_1 = v_2 = v_3 = 0$ is in the fiber.

Now choose a pair of deformations: $s = (A_s, a_s, B_s, b_s)$ with $A_s = \begin{pmatrix} -41 & -69 & 83 \\ -2 & -17 & 27 \\ -97 & -95 & 95 \end{pmatrix}$,

$$a_s = \begin{pmatrix} 87 \\ -51 \\ -98 \end{pmatrix}, B_s = \begin{pmatrix} 2 & -49 & 61 \\ -59 & -49 & -70 \\ -68 & -90 & 6 \end{pmatrix} \text{ and } b_s = \begin{pmatrix} 39 \\ -73 \\ 62 \end{pmatrix}. \text{ Compute the fiber}$$

of π_2 at s : $Y_s = \pi_2^{-1}(s)$. It turns out that $\dim(Y_s) = 3$ and that Y_s is a smooth variety. Let $d\pi_2$ be the differential of π_2 . Then the tangent space of Y_s at a smooth point is included into the kernel of $d\pi_2$. Therefore $\dim(\ker(d\pi_2)) \geq 3$.

Now a simple computation shows that $d\pi_2$ is surjective at the point defined by $\alpha = \beta = \gamma = x = y = z = 1, v_1 = v_2 = v_3 = 0$ and s . At this point as just mentioned, Y_s is smooth and thus its tangent space is three-dimensional. Therefore the dimension of Z , given by $\dim(\ker(d\pi_2)) + \text{rank}(d\pi_2)$, is greater or equal to $3 + \text{rank}(d\pi_2|_{\pi_2^{-1}(U)}) = 3 + 24 = 27$.

On the other hand, since there exist three-dimensional fibers, the dimension of Z must satisfy: $\dim(Z) \leq 3 + 24 = 27$ (see [20], chapter 4).

All together, we have proved that $\dim(Z) = 27$ and that the generic fiber is three-dimensional. \square

In the relation to the discussion above, one can check that in the neighborhood of each real solution, there infinitely other reals solutions. For examples provided that $(A, a), (B, b)$ is a solution, then $(\lambda A, a), (B, b)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ is also a solution, since the essential matrices remains unchanged up to a scale.

The practical consequence of this theorem is that one cannot hope to recover deformations from three images.

When the same deformation is repeated twice the system of equations is simplified.

But before we proceed more in depth, let us make the following observation. The deformations (A, a) and $(\lambda A, \lambda a)$ for $\lambda \neq 0$ produce the same image. Therefore one could conclude somewhat hastily that whatever the number of images, one can except just to recover the deformation modulo this equivalence. However observe that if multiples $(\lambda A, \lambda a)$ and $(\mu A, \mu a)$ of the same deformation are applied consecutively we get the following overall deformation $(\mu\lambda A^2, \mu\lambda Aa + \mu a)$ which is equivalent to $(A^2, Aa + a)$ only if $\mu\lambda = \mu$ or equivalently $\lambda = 1$. Therefore if exactly the same deformation is repeated twice, one can hope to be able to fully recover it. This is the conclusion the deeper analysis below will exhibit.

Consider the essential matrices E_{12} and E_{13} . E_{23} is the same as E_{12} . We compute A_0, a_0, C_0, c_0 as previously and for the actual deformation A, a there exist $\alpha \neq 0, v_1 \in \mathbb{R}^3, \gamma \neq 0$ and $v_3 \in \mathbb{R}^3$, such that:

$$\begin{cases} a = \alpha a_0 \\ A = 1/\alpha(A_0 + a_0 v_1^t) \\ Aa + a = \gamma c_0 \\ A^2 = 1/\gamma(C_0 + c_0 v_3^t) \end{cases}$$

This results in the following system of equation:

$$\begin{cases} (A_0 + a_0 v_1^t)a_0 + \alpha a_0 - \gamma c_0 & = 0 \\ \gamma(A_0 + a_0 v_1^t)^2 - \alpha^2(C_0 + c_0 v_3^t) & = 0 \\ \alpha x - 1 & = 0 \\ \gamma z - 1 & = 0 \end{cases} \quad (5)$$

Theorem 2. *For a generic affine deformation, repeated twice, one can recover this deformation from the three images. Moreover in a generic situation the solution is unique.*

Proof. Here X designates the sub-variety of \mathbb{A}^{10} defined by the system 5. Let $\alpha_0, \beta_0, x_0, z_0, v_{10}, v_{30}$ be a point on X . Let $a = \alpha_0 a_0, A = 1/\alpha_0(A_0 + a_0 v_{10}^t)$. Then we know that $Aa + a = \gamma_0 c_0$ and $A^2 = 1/\gamma_0(C_0 + c_0 v_{30}^t)$. Consider the variety Y defined by the following system:

$$\begin{cases} (A + a v_1'^t)a + \alpha' a - \gamma'(Aa + a) & = 0 \\ \gamma'(A + a v_1'^t)^2 - \alpha'^2(A^2 + (Aa + a)v_3'^t) & = 0 \\ \alpha' x' - 1 & = 0 \\ \gamma' z' - 1 & = 0 \end{cases} \quad (6)$$

The varieties X and Y are easily seen to be isomorphic. Indeed the following linear mapping: $(\alpha, \gamma, x, y, z, v_1^t, v_3^t) \mapsto (\alpha', \gamma', x', z', v_1'^t, v_3'^t) = (\alpha/\alpha_0, \gamma/\gamma_0, \alpha_0 x, \beta_0 y, \gamma_0 z, 1/\alpha_0^2(v_1^t - v_{10}^t), 1/\gamma_0^2(v_3^t - v_{30}^t))$ is an isomorphism from X and Y . Therefore $\dim(X) = \dim(Y)$.

As in theorem 1, let us consider the variety of all affine deformation $D \cong \mathbb{A}^{12}$, the variety $Z \subset \mathbb{A}^{10} \times D$ defined by the system 6 and eventually the projection: $\pi_2 : Z \longrightarrow D$.

Observe that here again the map π_2 is surjective, since given an affine deformation (A, a) , the system 6 always admits $\alpha' = \gamma' = x' = y' = 1$ and $v_1' = v_3' = 0$ as solution. This implies that $\dim(Z) \geq \dim(D) = 12$. As known, the dimension of any fiber of

π_2 is at least $\dim(Z) - \dim(D)$ and the minimal dimension of a fiber will exactly be $\dim(Z) - \dim(D)$. (see [20], chapter 4).

Now consider the deformation $s = (A_s, a_s)$ defined by $A_s = \begin{bmatrix} 27 & 99 & 92 \\ 8 & 29 & -31 \\ 69 & 44 & 67 \end{bmatrix}$ and $a_s = \begin{bmatrix} -32 \\ -74 \\ -4 \end{bmatrix}$. At this point, the fiber is zero dimensional, which shows that $\dim(Z) =$

12. Therefore for a generic deformation repeated twice, one can recover the deformation up to a finite fold ambiguity.

Now let us show that in a generic situation, there is exactly one answer. Observe that this is the case for the particular fiber considered above. Thus let $p \in Z$ be the unique point such that $\pi_2(p) = s$. As easily computed, the variety Z is smooth at this point and the projection π_2 is etale at p . Thus there exists U an open neighborhood of p in the Zariski topology, such that (i) Z is smooth on U and (ii) the projection π_2 is a diffeomorphism from U to $V = \pi_2(U)$. Therefore the map $\pi_{2|U} \rightarrow D$ is flat. Then all the fibers are isomorphic to the fiber over s and thus are made of a single point.

Moreover if there was another component of dimension 12, since the fiber over s is smooth, p cannot lie on the intersection of U and such a component. Thus the fiber over s should contain more than a single point, which is not the case. All together, we have proven that there is a single component of maximal dimension in Z and that the generic fiber over D is a singleton, or in other words, that for a generic situation, there is a single solution. \square

Here it is clear that the unique solution is real, since this is exactly the actual deformation that the points have undergone.

There are cases, other than two identical transformations, where the deformations are also solvable.

For example, when $B = \lambda A$ and $b = \mu a$ for unknown, non-zero scalars λ, μ . The system of equations 4 reduces to:

$$\begin{cases} \lambda(A_0 + a_0 v_1^t) a_0 + \alpha \mu a_0 - \gamma c_0 & = 0 \\ \lambda \gamma (A_0 + a_0 v_1^t)^2 - \alpha^2 (C_0 + c_0 v_3^t) & = 0 \end{cases} \quad (7)$$

This system is similar, but still different, than system 5. Of course, as previously, one has to add the two further equations $\alpha x - 1 = \gamma z - 1 = 0$. Now we shall prove the following result.

System 7 defines a discrete variety. As a consequence,

Theorem 3. *If (A, a) is the first deformation and $(\lambda A, \mu a)$ the second deformation ($\lambda \neq 0$ and $\mu \neq 0$), one can recover the two deformations and the structure up to a finite fold ambiguity.*

Proof. We proceed as in the the previous theorem. Here X designates the sub-variety of \mathbb{A}^{10} defined by system 7 (together with equations $\alpha x - 1 = \gamma z - 1 = 0$). Note that λ, μ are

not unknowns but parameters. Let $\alpha_0, \gamma_0, x_0, z_0, v_{10}, v_{30}$ be a point on X . Let $a = \alpha_0 a_0$, $A = 1/\alpha_0(A_0 + a_0 v_{10}^t)$. Then we know that $\lambda A a + \mu a = \gamma_0 c_0$ and $\lambda A^2 = 1/\gamma_0(C_0 + c_0 v_{30}^t)$. Consider the variety Y defined by the following system:

$$\begin{cases} \lambda(A + a v_1^t) a + \alpha' \mu a - \gamma'(\lambda A a + \mu a) & = 0 \\ \lambda \gamma'(A + a v_1^t)^2 - \alpha'^2(\lambda A^2 + (\lambda A a + \mu a) v_3^t) & = 0 \\ \alpha' x' - 1 & = 0 \\ \gamma' z' - 1 & = 0 \end{cases} \quad (8)$$

The varieties X and Y are isomorphic by the following linear mapping:

$$\begin{aligned} (\alpha, \gamma, x, y, z, v_1^t, v_3^t) &\mapsto (\alpha', \gamma', x', z', v_1^t, v_3^t) \\ &= (\alpha/\alpha_0, \gamma/\gamma_0, \alpha_0 x, \beta_0 y, \gamma_0 z, 1/\alpha_0^2(v_1^t - v_{10}^t), 1/\gamma_0^2(v_3^t - v_{30}^t)) \end{aligned}$$

is an isomorphism from X to Y . Therefore $\dim(X) = \dim(Y)$.

Again consider the variety D of all pairs $((A, a), (\lambda A, \mu a))$ (with $\lambda \neq 0$ and $\mu \neq 0$) where (A, a) is an affine deformation. Then $D \cong \mathbb{A}^{12} \times (\mathbb{A} \setminus \{0\})^2$.

Consider also as previously the variety $Z \subset \mathbb{A}^{10} \times D$ defined by the system 8 and eventually the projection: $\pi_2 : Z \longrightarrow D$.

Again observe that here again the map π_2 is surjective, since given an affine deformation (A, a) and two non-zero scalars λ, μ , the system 8 always admits $\alpha' = \gamma' = x' = y' = 1$ and $v_1' = v_3' = 0$ as solution. This implies that $\dim(Z) \geq \dim(D) = 14$. As mentioned above, the dimension of any fiber of π_2 is at least $\dim(Z) - \dim(D)$ and the minimal dimension of a fiber will exactly be $\dim(Z) - \dim(D)$.

Now consider the deformation $s = (A_s, a_s, \lambda_s, \mu_s)$ defined by $A_s = \begin{bmatrix} 27 & 99 & 92 \\ 8 & 29 & -31 \\ 69 & 44 & 67 \end{bmatrix}$,

$a_s = \begin{bmatrix} -32 \\ -74 \\ -4 \end{bmatrix}$, $\lambda_s = 1$ and $\mu_s = 1$. At this point, the fiber is zero dimensional,

which shows that $\dim(Z) = 14$. Therefore for a generic deformation and a generic point $(\lambda, \mu) \in (\mathbb{A} \setminus \{0\})^2$, one can recover the deformation and the scales λ, μ up to a finite fold ambiguity.

Here again, let us show that in a generic situation, there is exactly one answer. As previously, observe that this is the case for the particular fiber considered above. Thus let $p \in Z$ be the unique point such that $\pi_2(p) = s$. Again, as easily computed, the variety Z is smooth at this point and the projection π_2 is etale at p . Thus there exists U an open neighborhood of p in the Zariski topology, such that (i) Z is smooth on U and (ii) the projection π_2 is a diffeomorphism from U to $V = \pi_2(U)$. Therefore the map $\pi_{2|U} : U \longrightarrow D$ is flat. Then all the fibers are isomorphic to the fiber over s and thus are made of a single point.

Moreover if there was another component of dimension 14, since the fiber over s is smooth, p cannot lie on the intersection of U and such a component. Thus the fiber over s should contain more than a single point, which is not the case. All together, we have

proven that there is a single component of maximal dimension in Z and that the generic fiber over D is a singleton, or in other words, that for a generic situation, there is a single solution. \square

2.3 Shape Recovery

Once the deformation is known the shape before and after deformation is easily calculated. Indeed from the first image, the points are known up to scalar multiplication. From the second image, this scalar for each point is computed by linear means. The complicated part is to compute the deformation and this is our focus.

2.4 Critical Surface

Are there point configurations that do not allow the recovery of the essential matrix? It turns out the situation is similar to the classical case.

Assume that the projected points before and after deformation do not constrain the essential matrix uniquely. Therefore there exists more than one solution (homogeneous) to the system: $q_i^t E p_i = 0$. One is the correct solution $E_1 = [t]_{\times} A$, while another solution E_2 would have another decomposition. Therefore if there exists another solution, the points must satisfy:

$$q_i^t E_2 p_i = P_i^t \begin{bmatrix} A^t \\ t \end{bmatrix} E_2 [I; 0] P_i = 0$$

This means that the original points in space lie on a quadric, whose equation involves the affine motion that we are looking for. In this case, the recovery presents an additional layer of ambiguity. There exist several essential matrices and for each essential matrix, the corresponding affine motion is recovered up to the ambiguity described above.

2.5 Invariant Shape

The shape of a deforming object is by definition changing but there are descriptions that are invariant to the transformations, we shall show when these descriptions can be recovered from a sequence of images of a deforming object.

2.5.1 Equations

Let $P_0, P_1, P_2, P_3, \dots, P_{n-1}$ be $3d$ points in homogeneous coordinates with 1 as the last coordinate. Here and during all section 2.5, the points P_0, P_1, P_2, P_3 are assumed to define an affine basis of the three dimensional affine space \mathbb{A}^3 .

If a point P satisfies $P = \alpha P_0 + \beta P_1 + \gamma P_2 + (1 - \alpha - \beta - \gamma) P_3$, if it undergoes a 3D affine transformation, T , then,

$$TP = \alpha TP_0 + \beta TP_1 + \gamma TP_2 + (1 - \alpha - \beta - \gamma) TP_3.$$

Thus, (α, β, γ) is an affine invariant and α, β, γ and $1 - \alpha - \beta - \gamma$ are the affine invariant coordinates of P . In this section we aim at computing this affine invariant. The transformation itself is not recovered here. We deal with the simultaneous recovery of the transformation and the point coordinates in section 3.

The real advantage of this affine invariant is that it does not require camera calibration, while full recovery of deformation and structure requires it. On the other hand, the affine invariant description only provides structure up to an unknown affine deformation.

Let us write down the equations for the two image point sets $\{q_i\}$ and $\{q'_i\}$ of an affinely changing point set where $\begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}$ is the affine transformation and C the unknown camera matrix.

- For the first image, before the deformation, for each i , we have: $q_i \equiv C[\alpha_i P_0 + \beta_i P_1 + \gamma_i P_2 + (1 - \alpha_i - \beta_i - \gamma_i) P_3]$. Thus each image point gives two equations.
- After the deformation, in the second image, for each i , we have: $q'_i \equiv C \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} P_i \equiv C \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} [\alpha_i P_0 + \beta_i P_1 + \gamma_i P_2 + (1 - \alpha_i - \beta_i - \gamma_i) P_3]$. Again this yields two equations per point.

Basically the system has $2 \times 2n = 4n$ equations, where n is the number of points. As for the unknowns, there are $4 \times 3 + 3(n - 4) + 12 + 12 = 3n + 24$ unknowns namely $P_0, P_1, P_2, P_3, \{\alpha_i, \beta_i, \gamma_i\}_{i \geq 4}$, the camera matrix C and the affine transformation $\begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}$.

There are still ambiguities as, for any full rank V a 4×4 matrix with last row $[0, 0, 0, 1]$, $CP = (CV)(V^{-1}P)$, (new camera \times new points) and $C \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} P = (CV)(V^{-1} \begin{pmatrix} At \\ 01 \end{pmatrix} V)(V^{-1}P)$, (new camera \times new affine transformation \times new points). Since the only unknowns that are relevant are $\{\alpha_i, \beta_i, \gamma_i\}_{i \geq 4}$, we can assume $C = [I; 0]$, removing 12 unknowns. This formally makes the computation identical to the case of a calibrated camera, while calibration is not required here.

To make things more explicit, let us introduce new variables $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, such that $P_i = \lambda_i q_i$ for $0 \leq i \leq 3$. Then the equations can be written as follows:

$$q_i \wedge [\alpha_i(\lambda_0 q_0) + \beta_i(\lambda_1 q_1) + \gamma_i(\lambda_2 q_2) + \delta_i(\lambda_3 q_3)] = 0 \quad (9)$$

$$q'_j \wedge [A; t] \begin{bmatrix} \lambda_j q_j \\ 1 \end{bmatrix} = 0 \quad (10)$$

$$q'_i \wedge [A; t] \begin{bmatrix} \alpha_i(\lambda_0 q_0) + \beta_i(\lambda_1 q_1) + \gamma_i(\lambda_2 q_2) + \delta_i(\lambda_3 q_3) \\ 1 \end{bmatrix} = 0 \quad (11)$$

where $0 \leq j \leq 3$ and $4 \leq i \leq n - 1$ and $\delta_i = 1 - \alpha_i - \beta_i - \gamma_i$, \wedge being the cross product. These equations are homogeneous in $\{a_{ij}, t_k\}_{1 \leq i, j, k \leq 3}$ where $A = [a_{ij}]$ and $t = [t_1, t_2, t_3]^t$. Therefore they define an algebraic variety X in $\mathbb{P}^{11} \times \mathbb{A}^4 \times \mathbb{A}^{3(n-4)}$ (see [13, 8]), where

\mathbb{A}^k denotes the k -dimensional affine space. More precisely, since none of $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ should be zero, we need to compute in the localization of the polynomial ring with respect to each λ_i [7]. Here again, this is done by adding new variables $\{u_0, u_1, u_2, u_3\}$ and the equations:

$$\lambda_i \cdot u_i - 1 = 0 \quad (12)$$

Putting together all the equations (9,10,11,12) we get a variety $X \subset \mathbb{P}^{11} \times \mathbb{A}^8 \times \mathbb{A}^{3(n-4)}$. Since $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, u_0, u_1, u_2, u_3\}$ and $\{a_{ij}, t_k\}_{1 \leq i,j,k \leq 3}$ are not of interest, we eliminate them from the system and get a system involving only $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$. This is equivalent to projecting X over $\mathbb{A}^{3(n-4)}$, we are concerned with the case $n \geq 5$. The question now is: does this define a zero-dimensional variety or in other words can the affine invariant describing each point be computed up to a finite fold ambiguity. We address this question in the following subsection.

2.5.2 Dimension Analysis

The case of two images Let \overline{X} be the closure of X in $\mathbb{P}^{11} \times \mathbb{P}^8 \times \mathbb{P}^{3(n-4)}$. Let us denote $\pi_1 : \overline{X} \rightarrow \mathbb{P}^{11}$ the projection on the first space. The image Y_1 of \overline{X} by π_1 is precisely the variety of affine deformations that can explain the two images. If we have enough points $n \geq 7$, then by the beginning of section 2.2, its dimension is exactly 4. Moreover it is irreducible being isomorphic to $\mathbb{A}^3 \times \mathbb{A} \setminus \{0\}$.

Let us determine what are the fibers of π_1 . Given the deformation (A, t) , from equations (10), one can compute λ_j for $j = 0, 1, 2, 3$. Then from equations (9) and (11), we can compute $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$. Therefore, the fibers are all finite, such that X is irreducible and $\dim(X) = 4$. This can be rephrased very easily by saying that given the deformation, one can indeed recover the structure!

Now consider $\pi_3 : \overline{X} \rightarrow \mathbb{P}^{3(n-4)}$, the projection on the third space. Let Y_3 be the image $\pi_3(\overline{X})$. In order to answer the initial question: can the affine coordinates can be recovered from two images or equivalently one deformation, one need to determine if Y_3 is a discrete variety. In case it is, we could conclude that affine coordinates can be computed from two images up to a finite fold ambiguity.

We have that Y_3 is a discrete variety if and only if, the generic fiber of π_3 has at least dimension 4. However for a generic situation, this is not the case as explained below. Indeed, given $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$, equations (9) enable the computation of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, up to a scale factor. Similarly equations (10) and (11) enable in a generic situation, the recovery of $[A; t]$ up to a scale factor if $n \geq 7$. Therefore the generic fiber of π_3 is bi-dimensional, thus Y_3 cannot be a discrete variety. This yields the following conclusion:

Theorem 4. *Two images are not enough to recover the affine invariant coordinates of the points P_i for $4 \leq i \leq n-1$.*

One can wonder if some prior knowledge of the world can help to get a finite set of solutions. We show the following theorem:

Theorem 5. *If the 4 points $\{P_0, P_1, P_2, P_3\}$ are known and $n \geq 8$, then one can compute the affine invariant coordinates of the other points $\{P_i\}_{4 \leq i \leq n-1}$ from two images from a single non-calibrated camera, if the scene undergoes a general affine deformation and the points are in generic position.*

Proof. Since, $n \geq 8$ and the points are in generic position, one can compute the essential matrix, from the affine deformation can be extract up to a four fold ambiguity. The parametric representation of the set of acceptable deformations can be plugged in equations (10). Since the variables $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ are known, this allows to completely recover the deformation.

This situation can be described more formally as follows. Let Z be the variety in $\mathbb{P}^{11} \times \mathbb{A}^{3(n-4)}$ defined by equations (9,10,11). Note that in this case, we do not consider $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ as variables, since they are known and thus we do not include the localization equations (12) into our system.

Let $\pi_1 : \mathbb{P}^{11} \times \mathbb{A}^{3(n-4)}$ be the projection on the first factor. From the previous discussion, the closure $\pi_1(Z)$ is a finite set. The fiber is also finite as mentioned above. Therefore $\dim(Z) = 0$, that is Z is a finite too.

Therefore the projection of Z on the second of factor is necessary a finite variety, meaning that the affine invariant coordinate can be recovered up to a finite fold ambiguity. \square

In order to illustrate this approach, we proceeded to the following experiments. Consider the points $P_0 = [0, 0, 1, 1]^t$, $P_1 = [1, 0, 0, 1]^t$, $P_2 = [0, 1, 0, 1]^t$, $P_3 = [1, 2, 1, 1]^t$, $P_4 = [1, -1, -1, 1]^t$, $P_5 = [-2, 3, 1, 1]^t$, $P_6 = [2, -2, 3, 1]^t$, $P_7 = [-4, -2, 1, 1]^t$, $P_8 = [3, 2, 3, 1]^t$. The points are projected onto the first image to produce $\{q_i\}_{0 \leq i \leq 8}$. The variables $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are assigned their true values. The system of equations is formed and the deformation is eliminated, resulting in a system over $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq 8}$, which turns out to define a discrete variety. The computations were performed with Maple 16.

Three images In this section, we investigate the case of three images. Two configurations are possible, either the same deformation is repeated twice or two different deformations are performed.

The case of two distinct deformations is quickly dealt with, relying on theorem 1, one can prove the following result:

Theorem 6. *When the points undergo two unrelated generic affine deformations, the affine invariants $(\alpha_i, \beta_i, \gamma_i)$ cannot be computed up to a finite fold ambiguity.*

Proof. If we stack equations (9), (10), (11) and (12), the equations coming from a third image generated by a generic distinct affine deformations, we get a variety X embedded in $\mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{A}^8 \times \mathbb{A}^{3(n-4)}$. Projecting this variety over $\mathbb{P}^{11} \times \mathbb{P}^{11}$, we get the variety defined by the essential matrices. As known from theorem 1, this variety has strictly positive dimension. Then $\dim(X) > 0$ and the projection over $\mathbb{A}^{3(n-4)}$ will also have strictly positive dimension, since the generic fiber of this second projection is finite. \square

Therefore let us now turn our attention to the case where the points undergo the same deformations twice. In this context the unknowns are exactly the same as in the two images case: $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, $\{a_{ij}, t_k\}_{1 \leq i, j, k \leq 3}$, $t = [t_1, t_2, t_3]^t$ and $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$. The equations involved in this situation also contain those of the two images case and in addition equations similar to (10) and (11). Finally, with $P_i = \alpha_i(\lambda_0 q_0) + \beta_i(\lambda_1 q_1) + \gamma_i(\lambda_2 q_2) + \delta_i(\lambda_3 q_3)$, we get:

$$q_i \wedge P_i = 0 \quad (13)$$

$$q'_j \wedge [A; t] \begin{bmatrix} \lambda_j q_j \\ 1 \end{bmatrix} = 0 \quad (14)$$

$$q'_i \wedge [A; t] \begin{bmatrix} P_i \\ 1 \end{bmatrix} = 0 \quad (15)$$

$$q''_j \wedge [A^2; At + t] \begin{bmatrix} \lambda_j q_j \\ 1 \end{bmatrix} = 0 \quad (16)$$

$$q''_i \wedge [A^2; At + t] \begin{bmatrix} P_i \\ 1 \end{bmatrix} = 0 \quad (17)$$

As above, we add to these equations, the localization constraints expressed in 12. All together we get a variety $X \subset \mathbb{P}^{11} \times \mathbb{A}^8 \times \mathbb{A}^{3(n-4)}$. Again, we are interested in the projection of these variety into the factor $\mathbb{A}^{3(n-4)}$. However here we are in a position to prove the following result.

Theorem 7. *If the points undergo the same deformation twice, one can compute the affine invariant structure, i.e. $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$ up to a finite fold ambiguity from the three images.*

Proof. The proof is quite clear and works with the same scheme as the previous proofs. By eliminating from the equations the variables other than the affine deformations, we get exactly the same equations of the essential matrices. From theorem 2, we know that there is a single solution for the affine deformation. Then the other variables are uniquely determined. As a consequence, $\dim(X) = 0$. Therefore if one first eliminates the variables related to the deformation and the λ_i , a discrete variety for the affine invariant $(\alpha_i, \beta_i, \gamma_i)$ is left. \square

While this results provides a theoretical understanding, practically the task is quite cumbersome. One has to eliminate the following variables $(A, t, \lambda_0, \lambda_1, \lambda_2, \lambda_3, u_0, u_1, u_2, u_3)$ from the system made of previous equations, which leads to a system on $\{\alpha_i, \beta_i, \gamma_i\}_{4 \leq i \leq n-1}$ only. This latter system defines a discrete variety that can be theoretically computed.

3 Polynomial Deformations

In this section, we consider general polynomial deformations, the affine transformations treated above are a particular case. As previously, we shall assume that no degeneracies

occur, i.e. two points never collapse to a single point. This is ensured by assuming that the polynomial map is injective. Since we regard the deformation as a polynomial map from the complex three-dimensional affine space to itself, it is equivalent to say that it is bijective. This is known as the AxGrothendieck theorem. Of course, as previously the map itself has real coefficients in concrete situations.

3.1 Universal Matching Constraints

The essential matrix introduced in section 2.1 relies on the elimination of the 3D points from the projection equations before and after the deformation. For general polynomial deformations, such a process can also be carried out, leading to a polynomial matching constraint rather than a bilinear constraint.

In order to make things more concrete let us examine the equations. Let $\{q_i\}_i$ be the first image points. Let Φ be the deformation that the points undergo. Then in the second image, we get the following points: $q'_i \equiv \Phi(P_i)$, where $P_i = \alpha_i q_i$.

$$\Phi(X, Y, Z) = (\phi_1(X, Y, Z), \phi_2(X, Y, Z), \phi_3(X, Y, Z)),$$

where $\phi_1, \phi_2, \phi_3 \in \mathbb{R}[X, Y, Z]$ are polynomials. The system of equations is:

$$\alpha_i \cdot u_i - 1 = 0 \tag{18}$$

$$q'_i \wedge \Phi(\alpha_i q_i) = 0 \tag{19}$$

Here i indexes the points. Equations (18) are again localization equations, expressing that none of the α_i should be zero.

The matching polynomial is then obtained through the elimination of α and u from the 3 polynomial equations:

$$\begin{aligned} q'_x \Phi_3(\alpha q) &= \Phi_1(\alpha q) \\ q'_y \Phi_3(\alpha q) &= \Phi_2(\alpha q) \\ \alpha u &= 1 \end{aligned}$$

where all the unknown parameters of the deformation, q and q' are treated as variables. The elimination polynomial is treated as a k term polynomial in the variables of q and q' and can be recovered linearly using k point matches.

For example, if Φ is of the following form:

$$\begin{aligned} \Phi_1 &= a_x X + b_x Y + c_x Z + d_x \\ \Phi_2 &= a_y Y^2 + b_y X^2 + c_y Z^2 \\ \Phi_3 &= a_z Z^3 \end{aligned}$$

where $a_x, b_x, c_x, d_x, a_y, b_y, c_y, a_z$ are unknown parameters then the elimination polynomial

is:

$$\begin{aligned}
& c_y^3 q'_x + 3b_y c_y^2 q_x^2 q'_x + 3b_y^2 c_y q_x^4 q'_x + b_y^3 q_x^6 q'_x + 3a_y c_y^2 q'_x q_y^2 + 6a_y b_y c_y q_x^2 q'_x q_y^2 + 3a_y b_y^2 q_x^4 q'_x q_y^2 + 3a_y^2 c_y q'_x q_y^4 + \\
& 3a_y^2 b_y q_x^2 q'_x q_y^4 + a_y^3 q'_x q_y^6 - a_z c_x c_y q_y'^2 - a_x a_z c_y q_x q_y'^2 - a_z b_y c_x q_x^2 q_y'^2 - a_x a_z b_y q_x^3 q_y'^2 - \\
& a_z b_x c_y q_y q_y'^2 - a_z b_x b_y q_x^2 q_y q_y'^2 - a_y a_z c_x q_y^2 q_y'^2 - a_x a_y a_z q_x q_y^2 q_y'^2 - a_y a_z b_x q_y^3 q_y'^2 - a_z^2 d_x q_y'^3
\end{aligned}$$

which is a linear combination of the following 20 terms

$$\begin{aligned}
& q_x^6 q'_x, q_x^4 q'_x q_y^2, q_x^4 q'_x, q_x^3 q_y'^2, q_x^2 q'_x q_y^4, q_x^2 q'_x q_y^2, q_x^2 q_y q_y'^2, q_x^2 q'_x, q_x^2 q_y'^2, \\
& q_x q_y^2 q_y'^2, q_y'^2, q'_x q_y^6, q'_x q_y^4, q_y^3 q_y'^2, q'_x q_y^2, q_y^2 q_y'^2, q_y'^2, q'_x, q_y^3, q_y'^3
\end{aligned}$$

so the matching polynomial can be found with 20 matching points from 2 images.

3.2 Model Selection

The universal matching constraint exhibited in the previous section depends on the degree of the deformation. More precisely the number of monomials in q_i and q'_i for each matching pair (q_i, q'_i) is a function of the deformation degree and also of further assumptions on the deformation if there are available, as in the example above. These monomials must satisfies a linear equation: they form together the coordinates of points in a higher dimensional space.

Then computing the matching constraints is equivalent to finding the hyperplane that best fit all these points. This observation leads to a straight forward algorithm for model selection. A model, defined by the degree of the deformations and further assumption on the form of deformations if available, is rejected in the hyperplane assumption in the higher space is not acceptable. This can be implemented using some information criterion. One can refer for instance to [24] for further details.

3.3 Dimension Analysis

As we shall see again, at least three images are required. The camera is assumed to be calibrated described by the matrix $[I; 0]$.

We shall introduce a few notations. Let us denote \mathfrak{T}_d the variety of deformations of degree d . It is a mere affine space. Of course the following canonical injection holds: $\mathfrak{T}_k \subset \mathfrak{T}_d$ for $k < d$. Also let $\mathfrak{M}_k(\mathbf{q}, \mathbf{q}')$ be the subvariety of \mathfrak{T}_k defined by the matching constraints from the input $\mathbf{q} = (q_1, \dots, q_p)$ and $\mathbf{q}' = (q'_1, \dots, q'_p)$ where p is the number of terms appearing in the matching constraints. Of course p is polynomial function of k .

Consider a given deformation $\phi \in \mathfrak{M}_k(\mathbf{q}, \mathbf{q}')$. It can be decomposed into a sum of homogeneous deformations: $\phi = \psi_0 + \psi_1 + \dots + \psi_k$. Then $\psi_A = \psi_0 + \psi_1$ is an affine deformation. More schematically, we can write $\phi = \psi_A + \psi$ where $\psi = \psi_2 + \dots + \psi_k$.

Let $\mathbf{q}'_A = (q'_{1A}, \dots, q'_{pA})$ be the projected points after the deformation ψ_A . Note that they are not observed on the image since they are theoretically produced by the affine part

of the actual deformation. We have of course: $\psi_A \in \mathfrak{M}_1(\mathbf{q}, \mathbf{q}'_A)$. For any other affine deformation $\psi'_A \in \mathfrak{M}_1(\mathbf{q}, \mathbf{q}'_A)$, the deformation $\phi' = \psi'_A + \psi$ lies in $\mathfrak{M}_k(\mathbf{q}, \mathbf{q}')$. Therefore for a generic situation the following holds:

$$\dim(\mathfrak{M}_k(\mathbf{q}, \mathbf{q}')) \geq \dim(\mathfrak{M}_1(\mathbf{q}, \mathbf{q}'_A)) = 4$$

We are now in a position to formulate and prove the following theorem:

Theorem 8. *At least three images are required in order to recover the structure and deformation of a point set from a single calibrated camera.*

Proof. Now let $X \subset \mathfrak{T}_d \times \mathbb{A}^{2n}$ be the variety defined by the system of equations, where d denotes the degree of the selected model. Let $\pi_1 : X \rightarrow \mathfrak{T}_d$ and $\pi_2 : X \rightarrow \mathbb{A}^{2n}$ be the canonical projections. By [20], pp. 78-79, it is always true that $\dim(X) \geq \dim(\overline{\pi_1(X)})$, where $\overline{\pi_1(X)}$ denotes the Zariski closure of $\pi_1(X)$. With the previous notations, we have: $\overline{\pi_1(X)} = \mathfrak{M}_d(\mathbf{q}, \mathbf{q}')$. Hence $\dim(X) \geq 4 > 0$. Thus one cannot recover the deformation (and the structure) from two images only. \square

Therefore a third image must be considered. Following results from previous sections, in our setting, we considered the case where the scene undergoes the same deformation twice. Therefore one must add the following equations:

$$q''_i \wedge \Phi(\Phi(\alpha_i q_i)) = 0 \quad (20)$$

A simple counting procedure provides us the minimum number of points necessary to get a finite set of solutions.

Proposition 1. *Let d be the degree of the deformation and p be the dimension of the deformation space \mathfrak{T}_d . Under the assumption that the same deformation is repeated twice, at least $p/3$ input points are necessary to get a finite set of solutions.*

Proof. Equations (19,20) provide 4 independent equations per point. The localization equations (18) are n independent equations. There are $p + 2n$ unknowns. Therefore for a finite set of solutions, it is necessary that $4n + n \geq p + 2n$. Thus $n \geq p/3$. \square

In order to illustrate our considerations, we directly and simultaneously compute the structure and the deformation from equations (18), (19) and (20). More precisely, a set of points has been deformed with the following deformation:

$$\Phi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1/4X^2 + XY + X + Y + Z + 1 \\ 1/4Y^2 + YZ + X + Y + Z + 1 \\ 1/4Z^2 + XZ + X + Y + Z + 1 \end{pmatrix}$$

Both the deformation and the original set of points have been computed and as expected there is a single (real) solution. The computations have been performed using the minimal number of points presented in proposition 1 and were done with Maple.

4 Conclusion

We introduced a new problem in multiple-view geometry, i.e. the recovery of structure and deformation from a single perspective camera, where the deformation is either an affine or polynomial morphism of \mathbb{A}^3 and the camera is either calibrated or not. We showed several theoretical results and provided experiments in simple settings as a proof of concept. This paves the way for further theoretical and practical research about deformable configurations of points viewed from a monocular sequence.

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